Pergamon

Int. J. Heat Mass Transfer. Vol. 40, No. 15, pp. 3577–3580, 1997

Printed in Great Britain
0017–9310/97 $17.00+0.00

PII: S0017–9310(97)00005–7

The classical problem of convective heat transfer in laminar flow over a thin finite thickness plate with uniform temperature at the lower surface

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(*Received* 28 *March* 1996 *and in final form* 5 *December* 1996)

Abstract—In this paper we study the longitudinal heat conduction effects on the classical problem of forced laminar convection from a flat plate with an uniform temperature on the opposite surface. We found that small but noticeable differences when including the longitudinal heat conduction through the wall for the thermally thin wall regime as compared with the thermally thick wall regime solution. © 1997 Elsevier Science Ltd.

1. INTRODUCTION

The study of conjugate heat transfer between forced convection flows and conduction in walls is important due to the existence of coupled effects in practical heat transfer processes. Several works appear in the literature concerning the conjugate heat transfer process from a heated or cooled wall with finite thickness to a forced convective flow, where the temperature of the other surface is maintained at a constant and uniform temperature. Luikov [1] and Payvar [2] analyzed this conjugate problem. Luikov [1] developed two approximate solutions, one based on a differential analysis with low Prandtl number and the second based on a integral analysis with assumed polynomial forms for the velocity and temperature profiles. He concluded that for Brun numbers larger than 0.1, the thermal resistance of the wall can be neglected. The Brun number can be defined as the ratio α/ε^2, where α is the nondimensional longitudinal heat transfer parameter and ε is the aspect ratio of the plate (thickness to length). Payvar [2] used the Lighthill approximation [3] for large Prandtl numbers in order to obtain an integral equation which has been solved numerically. He obtained asymptotic solutions for large and small Brun numbers. This problem has been revised recently by Pozzi and Lupo [4] and Pop and Ingham [5]. In all the cited works only the transversal heat conduction through the plate has been considered and, therefore, neglected the longitudinal contribution.

The objective of the present work is to obtain analytically the overall heat transfer rates and the temperature profiles in the plate for the above-mentioned problem, with inclusion of the longitudinal heat conduction in the plate. The set of the governing equations are now elliptic and are to be solved using the Lighthill asymptotic approximation for large Prandtl numbers. The resulting energy equation for the flat plate depends fundamentally on two parameters: α and ε. These nondimensional parameters are to be defined in the following section. We use asymptotic techniques exploring analytically the limiting case of large values of α, with $\varepsilon \ll 1$ to be compared with the numerical solution.

2. ANALYSIS

A thin flat plate of length L and thickness h is placed parallel to a forced flow of a incompressible fluid with free-stream velocity U_∞ and temperature T_∞. The other surface of the plate is maintained at an uniform temperature T_w. The upper left corner of the plate coincides with the origin of a Cartesian coordinate system whose y axis points upward in the direction normal to the plate and its x axis points to the right in the plate's longitudinal direction.

The nondimensional governing equations are very well-known and can be found elsewhere [1, 2, 4, 5]. Introducing the following normalized variables

$$\theta = \frac{T - T_w}{T_\infty - T_w} \quad \chi = \frac{x}{L} \quad z = \frac{y}{h} \tag{1}$$

the nondimensional energy equation for the plate is given by the Laplace equation

$$\frac{\partial^2\theta}{\partial\chi^2}+\frac{1}{\varepsilon^2}\frac{\partial^2\theta}{\partial z^2}=0. \quad (2)$$

We assume, for simplicity, that both edges (leading and trailing edges) are adiabatic, which corresponds to the following boundary conditions

$$\frac{\partial\theta}{\partial\chi}=0 \quad \text{at } \chi=0 \quad \text{and} \quad \chi=1. \quad (3)$$

The boundary condition at the upper solid–fluid interface is obtained from continuity of the temperature and the heat flux. Using the asymptotic Lighthill approximation [3] valid for large Prandtl numbers compared with unity, this boundary condition can be written as

$$\left.\frac{\partial\theta}{\partial z}\right|_{z=0}=\frac{\varepsilon^2}{\alpha}\frac{1}{\sqrt{\chi}}\left[1-\theta_{\mathrm{lu}}-\int_0^{\chi}K_{\mathrm{u}}\frac{\mathrm{d}\theta'_{\mathrm{u}}}{\mathrm{d}\chi'}\mathrm{d}\chi'\right] \quad (4)$$

where

$$\alpha=\frac{1}{0.332}\frac{\lambda_{\mathrm{w}}}{\lambda}\frac{h}{L}\frac{1}{Re^{1/2}Pr^{1/3}} \quad \text{and} \quad \varepsilon=\frac{h}{L} \quad (5)$$

which represent the most relevant nondimensional parameters for this problem. Here, α is a nondimensional parameter which relates the solid longitudinal heat conduction to the convective heat towards the fluid and ε represents the aspect ratio of the plate assumed in this work to be much lower compared with unity. λ_{w} and λ represent the thermal conductivity of the wall material and the fluid, respectively. Re corresponds to the Reynolds number of the flow, $Re = U_\infty L/\nu$ and Pr to the Prandtl number, $Pr = \rho\nu c/\lambda$, where ν, ρ and c denote the kinematic viscosity, the density and the specific heat of the fluid, respectively. The Lighthill approximation gives accurate results also for values of the Prandtl numbers of order unity. The kernel of the integral in equation (4) is given by Ref. [3]

$$K_{\mathrm{u}}=\left(1-\left(\frac{\chi'}{\chi}\right)^{3/4}\right)^{-1/3} \quad (6)$$

where the subindex u denotes the upper interface and l denotes the leading edge. The other boundary condition is $\theta = 0$ at $z = -1$. The nondimensional global heat flux can be defined as

$$\overline{Nu}=\frac{1}{0.332Re^{1/2}Pr^{1/3}}\frac{\int_0^L q_{\mathrm{w}}(x)\,\mathrm{d}x}{\lambda(T_\infty-T_{\mathrm{w}})}=\frac{\alpha}{\varepsilon^2}\int_0^1\left.\frac{\partial\theta}{\partial z}\right|_{z=0}\mathrm{d}\chi. \quad (7)$$

For large values of α, that is $\alpha \gg 1$, the heat conducted by the plate is very large in all directions. Thus, no temperature gradients of importance arise in the wall. In this limit the nondimensional transversal variations of the wall temperature are very small of order ε^2/α. This limit is called the thermally thin wall regime. On the other hand, for $\alpha \ll 1$ (thermally thick wall regime), the longitudinal heat conduction through the solid can be neglected. In all the approaches published up to date on this problem, the longitudinal heat conduction through the wall has been neglected. The main objective of this work is to evaluate the longitudinal heat conduction through the solid in the thermally thin wall regime ($\alpha/\varepsilon^2 \gg 1$).

For the thermally thin wall regime, the longitudinal heat conduction can be important and must be retained in the analysis. However, in this problem the longitudinal temperature gradients cannot be large because of the boundary condition at the lower face of the plate. This is the reason why the thermally thick wall approximation gives very good results for the thermally thin wall regime. However, there are noticeable differences to be cleared in this work.

In the thermally thin wall regime the nondimensional temperature change in the transversal direction in the wall is very small. For this reason, also, the longitudinal change is of the same order of magnitude because the fixed boundary condition at the lower surface of the plate. In the asymptotic limit $\alpha \to \infty$, which is a regular limit, the nondimensional temperature at the wall is exactly zero everywhere in the plate. The problem then reduces to the case of the heat transfer process from an isothermal wall. In this limit, putting $\theta_{\mathrm{lu}} = \theta_{\mathrm{u}} = 0$ in equation (4) and integrating along the longitudinal direction in the wall we obtain $\lim_{\alpha\to\infty}\overline{Nu} = 2$. This is the same result if using the thermally thick wall approximation (no longitudinal heat conduction). The longitudinal heat conduction is not relevant because there is not any temperature gradient in the wall. However, for finite values of α, both solutions give different results. Without inclusion of the longitudinal heat conduction through the solid, equation (4) reduces to the following parameter-free equation

$$\theta_{\mathrm{u}}=\frac{1}{\sqrt{\xi}}\left[1-\theta_{\mathrm{lu}}-\int_0^{\xi}K_{\mathrm{u}}\frac{\mathrm{d}\theta'_{\mathrm{u}}}{\mathrm{d}\xi'}\mathrm{d}\xi'\right]$$
$$=-\frac{1}{\sqrt{\xi}}\int_0^{\xi}K_{\mathrm{u}}\frac{\mathrm{d}\theta'_{\mathrm{u}}}{\mathrm{d}\xi'}\mathrm{d}\xi' \quad (8)$$

where $\xi = \alpha^2\chi/\varepsilon^4$ and $\theta_{\mathrm{lu}} = 1$. This integral equation must be solved numerically. The overall reduced Nusselt number for this case is then given by

$$\overline{Nu}=\frac{\alpha}{\varepsilon^2}\int_0^1\theta_{\mathrm{u}}\,\mathrm{d}\chi=-\frac{\varepsilon^2}{\alpha}\int_0^{\alpha^2/\varepsilon^4}\frac{\mathrm{d}\xi}{\sqrt{\xi}}\int_0^{\xi}K_{\mathrm{u}}\frac{\mathrm{d}\theta'_{\mathrm{u}}}{\mathrm{d}\xi'}\mathrm{d}\xi'. \quad (9)$$

The asymptotic solution of equation (8) for large values of ξ is given by

$$\theta_{\mathrm{u}}\sim\frac{1}{\xi^{1/2}}-\frac{1.061}{\xi^{5/4}}+,\cdots, \quad \text{as } \xi\to\infty. \quad (10)$$

The overall Nusselt number is then described asymptotically by

$$\overline{Nu} \sim 2 - \frac{2.837}{\xi^{9/20}} \sim 2 - 2.837\left(\frac{\varepsilon^2}{\alpha}\right)^{9/10}$$

$$\text{for } \alpha \to \infty \quad \text{with fixed } \varepsilon. \quad (11)$$

This solution is surprisingly accurate. However, for the thermally thin wall regime, the behavior close to the leading edge is unacceptable. The longitudinal heat conduction term must be retained in a layer of order ε in χ, where the nondimensional temperature reach values of the order $\varepsilon^{3/2}/\alpha$. The reduced inner variables in this layer can then be defined by $\chi = \varepsilon\zeta$ and $\theta_u = (\varepsilon^{3/2}/\alpha)\varphi$. The inner universal problem to be solved is

$$\frac{\partial^2 \varphi}{\partial \zeta^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0 \quad (12)$$

with the following boundary conditions

$$\frac{\partial \varphi}{\partial \zeta} = 0 \quad \text{at } \zeta = 0, \quad \text{and} \quad \varphi \to 0 \quad \text{for } \zeta \to \infty \quad (13)$$

$$\varphi = 0 \quad \text{at } z = -1 \quad \frac{\partial \varphi}{\partial z} = \frac{1}{\zeta^{1/2}} \quad \text{at } z = 0. \quad (14)$$

The overall reduced Nusselt in this limit takes the form

$$\overline{Nu} \sim 2 - \frac{A\varepsilon^2}{\alpha} \quad (15)$$

where

$$A(\varepsilon) = \frac{2\varphi_{ul}}{\sqrt{\varepsilon}} + \int_0^{1/\varepsilon} \frac{1}{\sqrt{\zeta}} \int_0^{\zeta} K_u \frac{d\varphi'_u}{d\xi'} d\xi'. \quad (16)$$

Here, $\varphi_{ul} = \varphi(\zeta = 0, z = 0)$ and $\varphi_u = \varphi(\zeta, z = 0)$. Using the Fourier cosine transform technique, the solution to equations (12)–(14) is given by

$$\varphi_u(\zeta) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\tanh(k)\cos(k\zeta)\, dk}{k^{3/2}}$$

$$\text{with } \varphi_{ul} \doteq 3.0409. \quad (17)$$

For large values of ζ, the asymptotic solution is given by

$$\varphi_u(\zeta) \sim \frac{1}{\sqrt{\zeta}}, \quad \text{as } \zeta \to \infty. \quad (18)$$

The value of $A(\varepsilon)$ is a relatively smooth function of ε as shown in the following table:

ε	$A(\varepsilon)$
0.05	4.23423
0.1	4.12605
0.2	4.01173

3. RESULTS

Small, but noticeable, discrepancies are observed when using the thermally thick wall approximation for large values of α/ε^2. Figure 1 shows the overall reduced Nusselt number as a function of α/ε^2, using the numerical solution with the full Laplace equation and compared with the solution obtained using the thermally thick wall approximation, equation (9). The asymptotic solution for large values of α/ε^2 given by equation (11) is also plotted. We observe in this figure that both solutions diverge slightly one from each other for values of $\alpha/\varepsilon^2 \gg 1$. At the right wing of the figure, the overall reduced Nusselt number is slightly

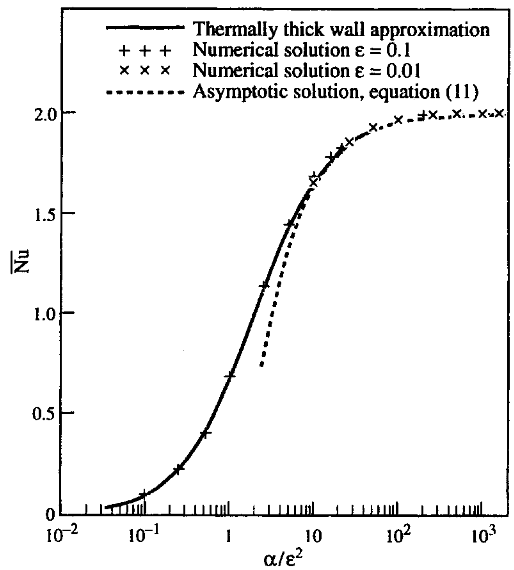

Fig. 1. Overall reduced Nusselt number as a function of α/ε^2, obtained using the thermally thick wall approximation and a numerical code with the full Laplace equation for the wall.

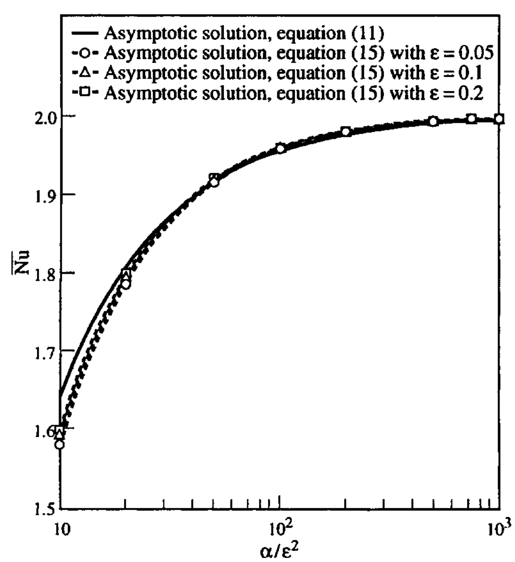

Fig. 2. Asymptotic solutions for the overall reduced Nusselt number as a function of α/ε^2 and different values of ε.

larger when including the longitudinal heat conduction. Figure 2 shows the asymptotic solutions for large values of α/ε^2, obtained using the thermally thick wall approximation (11) and that obtained using the full Laplace equation (15), with different values of ε. Both approximations give an excellent approximation to the solution. Including the longitudinal heat conduction through the plate give higher values of the overall reduced Nusselt number, which follows the same trend observed in Fig. 1. However, both approximations cross to each other at some position in α/ε^2. Of course at this position the asymptotic approximations are not more valid and what happens is that both asymptotic approximations merge, following the thermally thick wall approximation. As the value of ε decreases, both asymptotic approximations merge earlier. It is expected that in the limit of $\varepsilon \to 0$, both asymptotic approximations give the same results.

Acknowledgements—This work has been supported by a grant of DGAPA, UNAM IN107795.

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